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Hamiltonian structure of real Monge–Ampère equations

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Abstract. The variational principle for the real homogeneous Monge–Ampère equation in two dimensions is shown to contain three arbitrary functions of four variables. There exist two different specializations of this variational principle where the Lagrangian is degenerate and furthermore contains an arbitrary function of two variables. The Hamiltonian formulation of these degenerate Lagrangian systems requires the use of Dirac's theory of constraints. As in the case of most completely integrable systems the constraints are second class and Dirac brackets directly yield the Hamiltonian operators. Thus the real homogeneous Monge–Ampère equation in two dimensions admits two classes of infinitely many Hamiltonian operators, namely a family of local, as well as another family non-local Hamiltonian operators and symplectic 2-forms which depend on arbitrary functions of two variables. The simplest non-local Hamiltonian operator corresponds to the Kac–Moody algebra of vector fields and functions on the unit circle. Hamiltonian operators that belong to either class are compatible with each other but between classes there is only one compatible pair. In the case of real Monge–Ampère equations with constant right-hand side this compatible pair is the only pair of Hamiltonian operators that survives. Then the complete integrability of all these real Monge–Ampère equations follows by Magri's theorem. Some of the remarkable properties we have obtained for the Hamiltonian structure of the real homogeneous Monge–Ampère equation in two dimensions turn out to be generic to the real homogeneous Monge–Ampère equation and the geodesic flow for the complex homogeneous Monge–Ampère equation in arbitrary number of dimensions. Hence among all integrable nonlinear evolution equations in one space and one time dimension, the real homogeneous Monge–Ampère equation is distinguished as one that retains its character as an integrable system in multiple dimensions.

1. Introduction

Recently we had found [1] that the real homogeneous Monge–Ampère equation (RHMA) in two dimensions, hereafter to be referred to as RHMA_2, admits generalized Hamiltonian structure. In field theory this type of structure arises for degenerate Lagrangian systems where we must use Dirac's theory of constraints [2] to cast the problem in Hamiltonian form and finds its zenith in the theorem of Magri [3] for completely integrable bi-Hamiltonian systems. We refer the reader to [4] for an exposition of this subject. In this paper we shall show that RHMA and the geodesic flow for the complex homogeneous Monge–Ampère equation (CHMA) admit infinitely many Hamiltonian structures in arbitrary dimension. By Magri's theorem this result shows the complete integrability of RHMA and, in turn, RHMA provides the richest illustration of Magri's theorem.

It will be useful to first list the various homogeneous Monge–Ampère equations that we shall discuss in order to fix the notation. On a real manifold M of dimension n, RHMA_n is given by

$$\det u_{\alpha\beta} = 0 \qquad \alpha = 0, 1, \ldots, n-1 \tag{1}$$

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where

$$u_{\alpha\beta} \equiv \frac{\partial^2 u}{\partial x^\alpha \partial x^\beta}$$

is the matrix of second derivatives. We shall take $x^0 = t$ and write (1) as a first-order system of nonlinear evolution equations. Without loss of generality we may assume

$$\det u_{ab} \neq 0 \qquad a = 1, \ldots, n-1 \tag{2}$$

which is a statement of the non-degeracy of (1). In $1+1$ dimensions we have the simplest case of real Monge–Ampère equations:

$$\begin{aligned} & u_{tt}u_{xx} - {u_{tx}}^2 = -K \\ & K = \pm 1,\ 0 \end{aligned} \tag{3}$$

which is hyperbolic, elliptic, or homogeneous ($RHMA_2$), respectively. We shall show that two types of Hamiltonian operators exist for $RHMA_2$, a family of local (equation (29)), as well as non-local (equation (51)) operators, both of which contain an arbitrary function of two variables.

We shall start with a discussion of the variational principles underlying $RHMA_2$. The Lagrangian which results in the equations of motion for $RHMA_2$ contains three arbitrary functions of four variables. We shall concentrate on two qualitatively different classes of Lagrangians which are specializations of the master Lagrangian (10) for $RHMA_2$, cf equations (11) and (43) below, both of which contain arbitrary functions and which are furthermore degenerate. That is, the passage to a Hamiltonian formulation of these degenerate Lagrangian systems gives rise to second class constraints as in the case of most completely integrable systems [5]. The resulting two families of Hamiltonian operators, cf equations (29) and (51) contain all the information in the Dirac brackets of $RHMA_2$. These Hamiltonian operators are infinite in number as they contain arbitrary functions which make their first appearance in the Lagrangian formulation of $RHMA_2$ which, in turn, is related to its character as a universal field equation [6]. Any pair of local, or non-local Hamiltonian operators are compatible, however, a local Hamiltonian operator is not in general compatible with a non-local Hamiltonian operator with the exception of J_0 of (30) and J_1 given by (52). These two Hamiltonian operators determine part of the multi-Hamiltonian structure of the real Monge–Ampère equation (3) when the right-hand side is a non-zero constant.

We shall show that the Kac–Moody algebra that corresponds to the simplest non-local Hamiltonian operator (52) for $RHMA_2$ consists of the algebra of vector fields and functions on S^1. The family of local Hamiltonian operators can be brought to the standard form of a first-order operator by a Miura transformation. The recursion operator obtained by a composition of the simplest local and non-local Hamiltonian operators is Sheftel'-type [7], that is, it can be written as the square of a first-order operator which is a good recursion operator itself. Finally, we shall present the symplectic structure of $RHMA_2$ which is dual to these Hamiltonian structures. We shall also show that the symplectic 2-forms obtained in this way are the time components of the Witten–Zuckerman [8] 2-form that is the main geometrical object in the covariant formulation of symplectic structure.

In the theory of developable surfaces there is an equation which may be called Ur-$RHMA_2$

$$u_t\, u_x = k \tag{4}$$

where k is constant, as it is well known [9] that solutions of (4) must satisfy $RHMA_2$. We shall discuss the Hamiltonian structure of Ur-$RHMA_2$ in section 12 and show that the Hamiltonian operators appropriate to this equation consist of a scalar version of the Hamiltonian operators for RHMA_2 which lends further credence to its name.

We shall show that RHMA is an example of a nonlinear evolution equation that is integrable in an *arbitrary* number of dimensions, see [6] for other universal field equations with this property. In section 13 we shall find that some of the remarkable results we shall report on the Hamiltonian structure of RHMA_2 hold for RHMA_n as well. Hence RHMA_2 will serve as the prototype for the discussion of the symplectic structure of all real homogeneous Monge–Ampére equations. The Hamiltonian structure of the complex Monge–Ampère equation is quite different from that of RHMA and will not be considered in this paper. On the other hand the Hamiltonian structure of the geodesic flow defined by CHMA [10], which will be discussed in section 14, is very similar to that of RHMA.

2. First-order evolutionary form of RMA_2

For the Hamiltonian treatment of real Monge–Ampère equations we need to rewrite them as a system of nonlinear evolution equations which is first order in time. For RHMA_2 this is most conveniently accomplished by introducing the definitions

$$u_x = p \qquad u_t = q \tag{5}$$

and accordingly the equations of motion are either given by

$$u_t = q \qquad q_t = \frac{1}{u_{xx}}\left({q_x}^2 - K\right) \tag{6}$$

or

$$p_t = q_x \qquad q_t = \frac{1}{p_x}\left({q_x}^2 - K\right) \tag{7}$$

but we note that this split of (3) into either one of these pair of evolution equations is not unique. Thus we have necessarily introduced a degree of freedom over and above that present in (3) itself. We shall return to this point in section 5.3 on the Miura transformation.

Henceforth we shall use the notation $\{u^i;\ i = 1, 2\}$ for the first-order variables where u^1 will stand for either u, or p and $u^2 \equiv q$. Then the vector field defining the flow for (3) is given by

$$\boldsymbol{X}_u = q\frac{\partial}{\partial u} + \frac{q_x^2 - K}{u_{xx}}\frac{\partial}{\partial q} \qquad \boldsymbol{X}_p = q_x\frac{\partial}{\partial p} + \frac{q_x^2 - K}{p_x}\frac{\partial}{\partial q} \tag{8}$$

respectively. We shall cast the equations of motion following from (8) into the form of Hamilton's equations

$$u^i_t = \boldsymbol{X}(u^i) = \{u^i, H\} = J^{ik}\ \delta_k H \tag{9}$$

where the Hamiltonian operator J defining the Poisson bracket is a skew-symmetric matrix of differential operators satisfying the Jacobi identities and δ_i denotes the variational derivative with respect to u^i.

3. Variational formulation of RHMA_2

We shall find that the Hamiltonian operators for RHMA_2 are doubly infinite in number. The reason for the existence of such a multitude of Hamiltonian structures can be traced back to the fact that the variational principle (10) for the real homogeneous Monge–Ampère equation contains arbitrary functions. Our approach to the construction of the Hamiltonian operators as Dirac brackets appropriate to degenerate Lagrangian systems will make it manifest that the existence of arbitrary functions in Lagrangians for RHMA_2 is responsible for their appearance in the Hamiltonian operators.

It can be verified directly that the equations of motion following from the variational principle

$$\delta I = 0 \qquad = \int \mathcal{L}\, \mathrm{d}t\, \mathrm{d}x$$

with Lagrangian density

$$\begin{aligned} \mathcal{L} &= l_1\, u_{tt} + l_2\, u_{tx} + l_3\, u_{xx} \\ l_\alpha &= l_\alpha\left(u_t, u_x, \frac{u_{tt}}{u_{tx}}, \frac{u_{tx}}{u_{xx}}\right) \qquad \alpha = 1, 2, 3 \end{aligned} \tag{10}$$

yield RHMA_2 in the form of (3). There are three arbitrary functions of four variables in this Lagrangian. We have not explicitly indicated the possible dependence of l_α on the third combination of the ratio of second derivatives above as it is derivable from the others. It appears that this is the richest example of a variational principle for any nonlinear partial differential equation. Fairlie, Govaerts and Morozov [6] have considered the hierarchy of field equations where the equations of motion at any level are proportional to the Lagrangian at the next. They pointed out that this is a finite hierarchy ending in 'universal field equations'. RHMA_2 is precisely such a universal field equation and this is the reason why we should expect arbitrary functions in the Lagrangian (10).

In the variational formulation of (6) we shall consider various specializations of the arbitrary functions l_α in (10) which will still contain arbitrary functions of a more specialized variety. It is, however, important to remember that the enormous number of possibilities offered by the master Lagrangian (10) is the source of all the results we shall present below.

4. Local Dirac bracket for RHMA_2

A useful specialization of the Lagrangian density (10) that results in the first-order equations of motion (6) for RHMA_2 is given by

$$\mathcal{L}_\lambda = \lambda\, q_t - \lambda_{u_x}\, (u_t - q)\, q_x \tag{11}$$

where

$$\lambda = \lambda(u_x, q) \qquad \lambda_{u_x u_x} \neq 0 \tag{12}$$

is a twice differentiable arbitrary function of two variables. The canonical momenta appropriate to this Lagrangian are given by

$$\begin{aligned} \pi_1 \equiv \pi_u &= \frac{\partial \mathcal{L}_\lambda}{\partial u_t} = -\lambda_{u_x} q_x \\ \pi_2 \equiv \pi_q &= \frac{\partial \mathcal{L}_\lambda}{\partial q_t} = \lambda \end{aligned} \tag{13}$$

subject to the canonical Poisson bracket relations

$$\{\pi_i(x), u^k(y)\} = \delta^i_k\, \delta(x - y) \tag{14}$$

with all others vanishing. But the momenta (13) cannot be inverted for the velocities, or alternatively the Hessian vanishes

$$\det \left| \frac{\partial^2 \mathcal{L}_\lambda}{\partial u^i_t\ \partial u^k_t} \right| = 0 \tag{15}$$

and we have a degenerate Lagrangian system. Thus the passage to the Hamiltonian formulation of the Lagrangian (11) requires the use of Dirac's theory of constraints.

Following Dirac we introduce the two primary constraints that result from (13)

$$\begin{aligned}\phi_1 &= \pi_u + \lambda_{u_x} q_x \\ \phi_2 &= \pi_q - \lambda\end{aligned} \tag{16}$$

and calculate their Poisson brackets using the canonical Poisson bracket relations (14). The result

$$\begin{aligned}\{\phi_1(x), \phi_1(y)\} &= q_y\, \lambda_{u_y u_y}\, \delta_y(x-y) - q_x\, \lambda_{u_x u_x}\, \delta_x(y-x) \\ \{\phi_1(x), \phi_2(y)\} &= \lambda_{u_y}\, \delta_y(x-y) - \lambda_{u_x}\, \delta_x(y-x) - q_x\, \lambda_{q u_x}\, \delta(y-x) \\ \{\phi_2(x), \phi_2(y)\} &= 0\end{aligned} \tag{17}$$

shows that the constraints (16) are second class as they do not vanish modulo the constraints. This is a typical situation for integrable systems as in the example of KdV, or shallow water equations [5]. I am grateful to C A P Galvão [11] for pointing out to me that one should not simplify the Poisson brackets of the constraints (17) using the rules for manipulating distributions as such 'simplifications' often lead to incorrect Dirac brackets.

The total Dirac Hamiltonian is given by

$$H_{\mathrm{T}} = \int \left(\pi_i\, {u^i}_t - \mathcal{L} + c^i \phi_i\right)\, \mathrm{d}x \tag{18}$$

where c^i are Lagrange multipliers and summation over $i = 1, 2$ is implied. The condition that the constraints are maintained in time

$$\{\phi_i(x), H_{\mathrm{T}}\} = 0 \tag{19}$$

gives rise to no further constraints which would have been secondary constraints. Instead, using equations (17), we find that the Lagrange multipliers are determined from equations (19)

$$c^1 = q \qquad c^2 = \frac{{q_x}^2}{u_{xx}}$$

and they do not depend on the choice of the arbitrary function λ. This is expected since the constraints and therefore the total Hamiltonian is linear in the momenta, the correct equations of motion will result only if the Lagrange multipliers are simply the components of the vector field (8) for the flow. Finally from (18) we find

$$H_{\lambda,\mathrm{T}} = \int \left(q\, \pi_u + \frac{{q_x}^2}{u_{xx}}\, \pi_q - \frac{{q_x}^2\, \lambda}{u_{xx}}\right)\, \mathrm{d}x \tag{20}$$

for Dirac's total Hamiltonian. The check that with this total Hamiltonian all the equations of motion are satisfied is straightforward and we can summarize all of them in Hamilton's equations

$$\mathcal{A}_t = \{\mathcal{A}, H_{\lambda,\mathrm{T}}\} \tag{21}$$

where $\mathcal{A}$ is any functional of the canonical variables. There is, however, one further simplification that we can carry out because in Dirac's theory second class constraints hold as strong equations. This fact gives us the choice of eliminating the momenta from equations (13). Then we can write

$$\mathcal{H}_{\lambda,\mathrm{T}} = q\, q_x\, \lambda_{u_x} \tag{22}$$

for the total Dirac Hamiltonian density.

Given any two differentiable functionals of the canonical variables $\mathcal{A}$ and $\mathcal{B}$, the Dirac bracket is defined by

$$\{\mathcal{A}(x), \mathcal{B}(y)\}_{\mathrm{D}} = \{\mathcal{A}(x), \mathcal{B}(y)\} - \int \{\mathcal{A}(x), \phi_i(z)\} J^{ik}(z, w) \{\phi_k(w), \mathcal{B}(y)\} \, \mathrm{d}z \, \mathrm{d}w \tag{23}$$

where J^{ik} is the inverse of the matrix of Poisson brackets of the constraints. The definition of the inverse is simply

$$\int \{\phi_i(x), \phi_k(z)\} J^{kj}(z, y) \, \mathrm{d}z = \delta_i^j \, \delta(x - y) \tag{24}$$

which results in a set of differential equations for the entries of J^{ik}. Starting with the Poisson bracket relations (17) we find that equations (24) can be readily solved to yield

$$\begin{aligned}
J_\lambda^{11}(x, y) &= 0 \\
J_\lambda^{12}(x, y) &= -\frac{1}{\lambda_{u_x u_x} \, u_{xx}} \, \delta(x - y) \\
J_\lambda^{21}(x, y) &= \frac{1}{\lambda_{u_x u_x} \, u_{xx}} \, \delta(x - y) \\
J_\lambda^{22}(x, y) &= -\frac{2 \, q_x}{\lambda_{u_x u_x} \, {u_{xx}}^2} \, \delta_x(x - y) - \left(\frac{q_x}{\lambda_{u_x u_x} \, {u_{xx}}^2} \right)_x \delta(x - y)
\end{aligned} \tag{25}$$

for the inverse of (17). It will be convenient to rename the arbitrary function λ as μ according to

$$\lambda_{u_x u_x} \equiv \frac{1}{q} \, \mu_q \tag{26}$$

to simplify the calculations that will follow.

5. Local Hamiltonian operators for RHMA$_2$

The transition from the Dirac bracket to the Hamiltonian operator is given by

$$\{u^i(x), u^k(y)\}_{\mathrm{D}} = J^{ik}(x, y) \equiv J^{ik}(x) \delta(x - y) \tag{27}$$

and from (27) and (25) it follows that the Hamiltonian operator corresponding to the degenerate Lagrangian (11) is simply

$$J_\mu = \begin{pmatrix} 0 & \dfrac{q}{\mu_q \, u_{xx}} \\ -\dfrac{q}{\mu_q \, u_{xx}} & \dfrac{q \, q_x}{\mu_q \, u_{xx}^2} D_x + D_x \dfrac{q \, q_x}{\mu_q \, u_{xx}^2} \end{pmatrix} \tag{28}$$

which contains the arbitrary function μ of two variables. Two particular choices of this arbitrary function, namely $\mu = \frac{1}{2} q^2$ and $\mu = u_x q$ result in the bi-Hamiltonian structure of (3) reported in [1].

Under the change of variable $p = u_x$ the Hamiltonian operator (28) is transformed

$$J_\mu = \begin{pmatrix} 0 & D_x \dfrac{q}{\mu_q \, p_x} \\ \dfrac{q}{\mu_q \, p_x} D_x & \dfrac{q \, q_x}{\mu_q \, u_{xx}^2} D_x + D_x \dfrac{q \, q_x}{\mu_q \, u_{xx}^2} \end{pmatrix} \tag{29}$$

which is the form of the local Hamiltonian operator for RHMA_2 that we shall use henceforth. The most important element of the family of Hamiltonian operators in (29) is given by

$$J_0 = \begin{pmatrix} 0 & D_x \dfrac{1}{p_x} \\ \dfrac{1}{p_x} D_x & \dfrac{q_x}{p_x^2} D_x + D_x \dfrac{q_x}{p_x^2} \end{pmatrix} \tag{30}$$

which corresponds to the choice $\mu = \frac{1}{2} q^2$.

From the construction of the Dirac bracket in section 4 it is clear that the reason for the presence of the arbitrary function μ in the Hamiltonian operator (29) can be traced back to the degenerate Lagrangian (11) where it makes its first appearance as λ. In order to proceed we need to prove that (29) is indeed a Hamiltonian operator which requires a check of the Jacobi identities. However, this also follows from the fact that (29) is derived from the Dirac bracket (25) for which we have a general proof of the Jacobi identities [12].

5.1. Jacobi identities

The Hamiltonian operator is a bi-vector which defines the Poisson bracket. The skew-symmetry of the operator (29) is manifest. In order to verify that a given bi-vector is Hamiltonian we must verify that it satisfies the tri-vector Jacobi identities. Thus following Olver [4] we introduce an arbitrary basis of tangent vectors Θ which are then conveniently manipulated according to the rules of exterior calculus. The Jacobi identities are given by the compact expression

$$L\,\delta I = 0 \qquad \text{(mod. div.)} \tag{31}$$

where

$$L = J\,\Theta \qquad I = \tfrac{1}{2}\,\Theta^T \wedge J\Theta \tag{32}$$

and δ denotes the variational derivative. The vanishing of the tri-vector (31) modulo a divergence is equivalent to the satisfaction of the Jacobi identities.

For the Hamiltonian operator (29) we have a two component system and introducing the basis

$$\Theta = \begin{pmatrix} \eta \\ \theta \end{pmatrix}$$

from equations (32) we have

$$L_\mu = \begin{pmatrix} \left(\dfrac{q}{\mu_q p_x}\right)_x \theta + \dfrac{q}{\mu_q p_x} \theta_x \\ \dfrac{q}{\mu_q p_x} \eta_x + 2 \dfrac{q q_x}{\mu_q p_x^2} \theta_x + \left(\dfrac{q q_x}{\mu_q p_x^2}\right)_x \theta \end{pmatrix}$$
$$I_\mu = \frac{q}{\mu_q p_x} \left(\theta \wedge \eta_x + \frac{q_x}{p_x} \theta \wedge \theta_x\right). \tag{33}$$

To form the required expression in (31) we first calculate the variational derivatives

$$\begin{aligned}\frac{\delta I_\mu}{\delta p} = & \left[\frac{q_x}{\mu_q p_x^2} - \frac{q}{\mu_q^2 p_x^2}\left(\mu_{qq} q_x + 2\mu_{pq} p_x\right) - 2\frac{q p_{xx}}{\mu_q p_x^3}\right]\theta\wedge\eta_x \\ & + \left[2\frac{q_x^2 + q q_{xx}}{\mu_q p_x^3} - 2\frac{\mu_{qq} q q_x^2}{\mu_q^2 p_x^3} - 3\frac{\mu_{pq} q q_x}{\mu_q^2 p_x^2} - 6\frac{q q_x p_{xx}}{\mu_q p_x^4}\right]\theta\wedge\theta_x \\ & + \frac{q}{\mu_q p_x^2}\left(\theta_x\wedge\eta_x + \theta\wedge\eta_{xx}\right) + 2\frac{q q_x}{\mu_q p_x^3}\,\theta\wedge\theta_{xx} \end{aligned} \qquad (34)$$

$$\frac{\delta I_\mu}{\delta q} = \left(\frac{1}{\mu_q p_x} - \frac{q\mu_{qq}}{\mu_q^2 p_x}\right)\theta\wedge\eta_x + \left(\frac{q\mu_{pq}}{\mu_q^2 p_x} + 2\frac{q p_{xx}}{\mu_q p_x^3}\right)\theta\wedge\theta_x - \frac{q}{\mu_q p_x^2}\,\theta\wedge\theta_{xx}$$

and take their exterior product with L_μ. The result

$$L_\mu\ \delta I_\mu = -\left(\frac{q^2}{\mu_q^2\ p_x^3}\ \theta\wedge\theta_x\wedge\eta_x\right)_x$$

is a total derivative so that the Jacobi identities are satisfied.

5.2. Hamilton's equations

It can be directly verified that

$$\mathcal{H}_\mu = \mu\ p_x \qquad (35)$$

is conserved for the flow (8) and this is the Hamiltonian density appropriate to the operator (29). That is, the equations of motion (8) are cast into the form of Hamilton's equations (9) with J_μ and H_μ. But since μ is an arbitrary function of p and q, given any other differentiable function $\nu = \nu(p, q)$ we have

$$J_\mu\ \delta H_\mu = J_\nu\ \delta H_\nu \qquad (36)$$

which is a statement of the Lenard–Magri recursion relation. Thus we have another expression of the fact that there exist infinitely many Hamiltonian operators for RHMA_2. The Casimir density for the Hamiltonian operator (29) is given by

$$\mathcal{C}_\mu = \sigma\ p_x \qquad (37)$$

where

$$\sigma_q = \frac{1}{q}\mu_q \qquad (38)$$

since it can be readily verified that $J_\mu \delta C_\mu = 0$. The proof of the Jacobi identities for (29) in section 5.1 is for arbitrary μ, hence all local Hamiltonian operators J_μ are compatible with each other.

5.3. The Miura transformation

The remarkable feature of the Hamiltonian operator (29) is the existence of μ, an arbitrary function of two variables. We have already remarked that for the Hamiltonian formulation of RHMA_2 we need to start with a pair of nonlinear evolution equations and therefore had to use the equations of motion defined by the vector field (8) rather than equation (3) itself. This introduces an extra degree of freedom as the definitions of p, q cannot be unique. It may appear that this extra degree of freedom could be responsible for the existence of the arbitrary function μ in the Hamiltonian operator (29). This is not the case. We have shown that the reason for the appearance of this arbitrary function in the Hamiltonian operator can be traced back to the existence of arbitrary functions in the master Lagrangian. Nevertheless, there is certainly an extra degree of freedom available in the definition of the first-order dependent variables for RHMA_2 and it is possible to exploit it for various purposes. In fact by a redefinition of q the Hamiltonian operator (29) can be brought to the standard form of a first-order operator with constant coefficients.

In place of (7) we may use the following definitions for the auxiliary variables p, q

$$u_x = p \qquad u_t = Q\left(p, \frac{q}{p_x}\right) \tag{39}$$

where Q is a differentiable function of its arguments. The equations of motion are now given by

$$p_t = Q_x \qquad q_t = \left(\frac{q}{p_x} Q_x\right)_x \tag{40}$$

which are already in standard Hamiltonian form. That is, we have

$$\begin{pmatrix} p_t \\ q_t \end{pmatrix} = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix} \begin{pmatrix} \delta_p \\ \delta_q \end{pmatrix} H \tag{41}$$

with the familiar first-order Hamiltonian operator and the Hamiltonian density is given by

$$\mathcal{H} = h\left(p, \frac{q}{p_x}\right) p_x \tag{42}$$

where h is related to Q through $\mathcal{H}_q = Q$. The transformation of variables obtained by a comparison of (39) and (5) is a Miura transformation as it brings the Hamiltonian operator to the canonical form (41) of Gardner–Zakharov–Fadeev [13] with constant coefficients.

6. Non-local Dirac bracket for RHMA_2

There is another family of Hamiltonian operators for equations (6) which falls outside the class of Hamiltonian operators in (29). Its origin can be traced back to the fact that

$$\mathcal{L}_\kappa = \kappa\, p_x + \kappa_r\,(p_t - q_x) \tag{43}$$

with κ an arbitrary function of two variables

$$\kappa = \kappa\,(p, r) \qquad r \equiv \frac{q_x}{p_x} \qquad \kappa_{rr} \neq 0 \tag{44}$$

is a new Lagrangian for the system (6). This is another specialization of the master Lagrangian (10) which is also degenerate and the passage to its Hamiltonian formulation again requires the use of Dirac's theory.

Starting with the Lagrangian (43) the momenta are given by

$$\begin{aligned} \Pi_1 \equiv \Pi_p &= \frac{\partial \mathcal{L}_\kappa}{\partial p_t} = \kappa_r \\ \Pi_2 \equiv \Pi_q &= \frac{\partial \mathcal{L}_\kappa}{\partial q_t} = 0 \end{aligned} \tag{45}$$

which cannot be inverted for the velocities. So we introduce the primary constraints

$$\begin{aligned} \Phi_1 &= \Pi_p - \kappa_r \\ \Phi_2 &= \Pi_q \end{aligned} \tag{46}$$

and using the canonical Poisson bracket relations (14) we find

$$
\begin{aligned}
\{\Phi_1(x), \Phi_1(y)\} &= \kappa_{rr}(x)\,\frac{q_x}{p_x^{\,2}}\,\delta_x(x-y) - \kappa_{rr}(y)\,\frac{q_y}{p_y^{\,2}}\,\delta_y(y-x) \\
\{\Phi_1(x), \Phi_2(y)\} &= \kappa_{rr}(x)\,\frac{1}{p_x}\,\delta_y(x-y) \\
\{\Phi_2(x), \Phi_2(y)\} &= 0
\end{aligned}
\tag{47}
$$

which show that the constraints are once again second class. The total Hamiltonian consists of a sum of the free Hamiltonian with a linear combination of the constraints as in (18) and the requirement that the constraints (46) are maintained in time, cf equations (19), does not lead to secondary constraints but instead determines the Lagrange multipliers

$$
c^1 = q_x \qquad c^2 = \frac{q_x^{\,2}}{p_x}
$$

which also do not depend on the arbitrary function κ. Using this information we find that

$$
H_{\kappa\,T} = \int \left(q_x\,\Pi_p + \frac{q_x^{\,2}}{p_x}\,\Pi_q - \kappa\,p_x \right) \mathrm{d}x \tag{48}
$$

is the total Hamiltonian and the equations of motion are given by (21). Once again we can simplify the total Hamiltonian using the fact that second-class constraints (46) hold as strong equations and we find

$$
\mathcal{H}_{\kappa\,T} = \kappa_r\,q_x - \kappa\,p_x \tag{49}
$$

for the Dirac Hamiltonian density.

For the Dirac bracket we need the inverse of the Poisson brackets of the constraints (47). The solution of equations (24) is given by

$$
\begin{aligned}
J_\kappa^{11}(x, y) &= 0 \\
J_\kappa^{12}(x, y) &= -\frac{p_x}{\kappa_{rr}(x)}\,\theta(x-y) \\
J_\kappa^{21}(x, y) &= -\delta(x-y)\int^x \frac{p_\xi}{\kappa_{rr}(\xi)}\,\mathrm{d}\xi \\
J_\kappa^{22}(x, y) &= -\frac{q_x}{\kappa_{rr}(x)}\,\theta(x-y) - 2\,\delta(x-y)\int^x \frac{q_\xi}{\kappa_{rr}(\xi)}\,\mathrm{d}\xi
\end{aligned}
\tag{50}
$$

where θ is the Heaviside unit step function. The Dirac bracket for the Lagrangian (43) now follows directly from (23).

7. Non-local Hamiltonian operators for RHMA$_2$

The correspondence (27) enables us to express the result (50) for the Dirac bracket of the degenerate Lagrangian (43) in the form of the Hamiltonian operator

$$
J_\kappa = \begin{pmatrix} 0 & \dfrac{p_x}{\kappa_{rr}} D_x^{-1} \\ D_x^{-1}\dfrac{p_x}{\kappa_{rr}} & \dfrac{q_x}{\kappa_{rr}} D_x^{-1} + D_x^{-1}\dfrac{q_x}{\kappa_{rr}} \end{pmatrix} \tag{51}
$$

where D_x^{-1} is the inverse of D_x. We refer the reader to [14] for the definition and properties of D_x^{-1}; in particular

$$D_x^{-1} f = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right) f(\xi)\, \mathrm{d}\xi$$

and the integrals are taken in the principal value sense. This Hamiltonian operator is therefore non-local. Once again, the appearance of the arbitrary function κ in the Hamiltonian operator (51) is a consequence of the existence of arbitrary functions in the master Lagrangian (10).

The most important member of the family of non-local Hamiltonian operators (51) is given by

$$J_1 = \begin{pmatrix} 0 & p_x D_x^{-1} \\ D_x^{-1} p_x & q_x D_x^{-1} + D_x^{-1} q_x \end{pmatrix} \tag{52}$$

which has a linear dependence on p_x and q_x. It results from the Lagrangian

$$\mathcal{L}_1 = \frac{1}{p_x}\left(p_t\, q_x - \frac{1}{2} {q_x}^2\right) \tag{53}$$

with the simplest choice of arbitrary function $\kappa = \frac{1}{2} r^2$. It will also be useful to rewrite the operator (52) for the system of variables consisting of u and q

$$J_1 = \begin{pmatrix} 0 & D_x^{-1} u_{xx} D_x^{-1} \\ -D_x^{-1} u_{xx} D_x^{-1} & q_x D_x^{-1} + D_x^{-1} q_x \end{pmatrix} \tag{54}$$

which follows from the change of variable $p = u_x$.

In section 7.3 we shall find that the non-local Hamiltonian operator (52) has a natural interpretation, namely the Kac–Moody algebra corresponding to this operator is the algebra of vector fields and functions on the unit circle. This operator is also distinguished as the only element of the family of non-local Hamiltonian operators that survives in the case of the real Monge–Ampère equation with constant right-hand side.

7.1. Jacobi identities

The proof of the Jacobi identities for J_κ of (51) proceeds along the general lines indicated in section 5.1 but the properties of D_x^{-1} must be carefully considered. For ease of writing we let $\rho \equiv (\kappa_{rr})^{-1}$ and from (51) we have

$$\begin{aligned} L_\kappa &= \begin{pmatrix} \rho p_x D_x^{-1}\theta \\ D_x^{-1}(\rho p_x \eta) + \rho q_x D_x^{-1}\theta + D_x^{-1}(\rho q_x \theta) \end{pmatrix} \\ I_\kappa &= \rho p_x \eta \wedge D_x^{-1}\theta + \rho q_x \theta \wedge D_x^{-1}\theta\,. \end{aligned} \tag{55}$$

The variational derivatives of I_κ are given by

$$\begin{aligned} \frac{\delta I_\kappa}{\delta p} &= (r\rho_r - \rho)\left(\eta_x \wedge D_x^{-1}\theta + \eta \wedge \theta\right) + r\rho_{rx}\eta \wedge D_x^{-1}\theta \\ &\quad + \left(r^2\rho_r\right)_x \theta \wedge D_x^{-1}\theta - r^2\rho_r\theta_x \wedge D_x^{-1}\theta \\ \frac{\delta I_\kappa}{\delta q} &= -\rho_{rx}\eta \wedge D_x^{-1}\theta - \rho_r\left(\eta_x \wedge D_x^{-1}\theta + \eta \wedge \theta\right) \\ &\quad - (\rho + r\rho_r)\,\theta_x \wedge D_x^{-1}\theta - (\rho + r\rho_r)_x\,\theta \wedge D_x^{-1}\theta \end{aligned} \tag{56}$$

which results in a total derivative for the tri-linear form (31)

$$L_\kappa \delta I_\kappa = -\left\{D_x^{-1}(p_x\eta + q_x\theta) \wedge [\rho\eta + (\rho + r\rho_r)\,\theta] \wedge D_x^{-1}\theta\right\}_x$$

completing the proof of the Jacobi identities.

7.2. Hamilton's equations

The Hamiltonian function H_κ appropriate to the non-local Hamiltonian operator (51) is given by a local expression which is simply the integral of the total Hamiltonian density of Dirac in (49) and once again we have the Lenard–Magri recursion relation

$$J_\kappa \;\delta H_\kappa = J_\iota \;\delta H_\iota \tag{57}$$

where ι is another arbitrary function of p, r. All non-local Hamiltonian operators are compatible with each other.

7.3. The Kac–Moody algebra

Hamiltonian operators associated with integrable nonlinear evolution equations give rise to Kac–Moody (KM) algebras [15]. Two compatible Hamiltonian operators actually yield an infinite hierarchy of KM algebras. There is an explicit algorithm for the construction of KM algebras from the Hamiltonian operator which is essentially based on Fourier analysis. Since the non-local operator (52) depends linearly on p_x and q_x Fourier analysis makes sense and the operator (52) is suitable for consideration as the backbone of a possible KM algebra. The nonlinearities in the local operator present a formidable obstacle to any similar discussion of (29). Using

$$p(x) = \frac{1}{2\pi}\int \frac{1}{n}\mathcal{P}_n\,\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}n \qquad q(x) = \frac{1}{2\pi}\int \frac{1}{m}\mathcal{Q}_m\,\mathrm{e}^{\mathrm{i}mx}\,\mathrm{d}m$$

we find that the KM algebra appropriate to the operator (52) is given by

$$\begin{aligned}
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \\
&[\mathcal{P}_m, \mathcal{Q}_n] = m\,\mathcal{P}_{m+n} \\
&[\mathcal{Q}_m, \mathcal{Q}_n] = (m-n)\,\mathcal{Q}_{m+n}
\end{aligned} \tag{58}$$

which can be recognized as the algebra of functions and vector fields on S^1. Thus we can use the representation

$$\mathcal{P}_n = z^{-n} \qquad \mathcal{Q}_m = z^{-m+1}\,\frac{\mathrm{d}}{\mathrm{d}z}$$

for the algebra underlying the simplest non-local Hamiltonian operator (52).

7.4. Compatibility

Two Hamiltonian operators are compatible if their linear combination with constant coefficients is also a Hamiltonian operator. In the case where one of the Hamiltonian operators belongs to the class of (29) and the other one to (51), the check of compatibility requires that

$$C_{\mu\kappa} = L_\mu\,\delta I_\kappa + L_\kappa\,\delta I_\mu = 0 \tag{59}$$

modulo a divergence. It turns out that only the simplest local and non-local operators J_0 and J_1 are compatible. From equations (33), (34), (55) and (56) it follows that

$$C_{01} = \left(\frac{1}{p_x^2}\, \theta \wedge \left[(q_x\theta_x + p_x\eta_x) \wedge D_x^{-1}\theta + D_x^{-1}(p_x\eta + q_x\theta) \wedge \theta_x \right] \right)_x$$

which establishes the compatibility of J_0 and J_1. In all other cases local and non-local Hamiltonian operators are incompatible.

8. Recursion operators

Since there are infinitely many compatible Hamiltonian operators of both local (29) as well as non-local (51) variety, there exist infinitely many opportunities for constructing recursion operators of either type or both. In the first category we have

$$\mathcal{R}_{\mu\nu} = J_\mu J_\nu^{-1} = \begin{pmatrix} D_x \dfrac{\nu_q}{\mu_q} D_x^{-1} & 0 \\ \left(\dfrac{\nu_q}{\mu_q}\right)_x \dfrac{q_x}{p_x} D_x^{-1} & \dfrac{\nu_q}{\mu_q} \end{pmatrix} \tag{60}$$

which generalizes our earlier result [1], while in the latter category we find

$$\mathcal{R}_{\kappa\iota} = J_\kappa J_\iota^{-1} = \begin{pmatrix} \dfrac{\iota_{rr}}{\kappa_{rr}} & 0 \\ D_x^{-1}\left[\dfrac{q_x}{p_x}\left(\dfrac{\iota_{rr}}{\kappa_{rr}}\right)_x\right] & D_x^{-1}\dfrac{\iota_{rr}}{\kappa_{rr}} D_x \end{pmatrix}. \tag{61}$$

These recursion operators satisfy the Lax equation

$$\mathcal{R}_t = [\mathcal{R}, \mathcal{A}] \tag{62}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & D_x \\ -\dfrac{q_x^2}{p_x^2} D_x & 2\dfrac{q_x}{p_x} D_x \end{pmatrix} \tag{63}$$

is obtained from the Frechét derivative of the flow (8).

However, there is also the possibility of constructing recursion operators by the composition of Hamiltonian operators of both local and non-local types provided they are compatible. We have found that J_0 and J_1 form a compatible pair. Thus we are led to the recursion operator

$$\mathcal{R} = J_0 J_1^{-1} = \begin{pmatrix} D_x \dfrac{1}{p_x} D_x \dfrac{1}{p_x} & 0 \\ -\dfrac{q_{xx}}{p_x^3} D_x - D_x \dfrac{q_{xx}}{p_x^3} & \dfrac{1}{p_x} D_x \dfrac{1}{p_x} D_x \end{pmatrix} \tag{64}$$

and we find that a remarkable factorization takes place

$$\mathcal{R} = \mathcal{E}\mathcal{E} \tag{65}$$

where

$$\mathcal{E} = \begin{pmatrix} D_x \dfrac{1}{p_x} & 0 \\ -\dfrac{q_{xx}}{p_x^2} & \dfrac{1}{p_x} D_x \end{pmatrix} \tag{66}$$

is a first-order operator. This situation is familiar from third-order Hamiltonian operators for two component equations of hydrodynamic type [7]. The recursion operator obtained from a composition of third and first-order Hamiltonian operators factorizes to yield the first-order recursion operator of Sheftel'. Similar considerations suggest that $\mathcal{E}$ itself is a good recursion operator. It is readily verified that $\mathcal{E}$ as well as its inverse

$$\mathcal{E}^{-1} = \begin{pmatrix} p_x D_x^{-1} & 0 \\ q_x D_x^{-1} - D_x^{-1} q_x & D_x^{-1} p_x \end{pmatrix} \tag{67}$$

satisfy equation (62) and are therefore good recursion operators. Finally, we should remark that the recursion operators (60) and (61) can also be factorized, however, unlike the case of (65), the factors consist of different operators and each one is not a recursion operator.

8.1. Infinite sequences of Hamiltonians

There are several infinite families of conserved quantities that one can obtain by repeated application of the recursion operators we have presented in section 8. Corresponding to the recursion operator $\mathcal{E}$ we have $\mathcal{F}$ which is defined by

$$\mathcal{S} = J_0{}^{-1} J_1 = \mathcal{F}\mathcal{F} \tag{68}$$

which yields infinite sequences of gradients of conserved quantities. We find that $\mathcal{F}$ is given by

$$\mathcal{F} = \begin{pmatrix} D_x^{-1} p_x & D_x^{-1} q_x - q_x D_x^{-1} \\ 0 & p_x D_x^{-1} \end{pmatrix} \tag{69}$$

and a simple example of the infinite sequence of conserved quantities that it generates is the following:

$$p\, q_x \xrightarrow{\mathcal{F}} \cdots \xrightarrow{\mathcal{F}} \frac{1}{n!}\, p^n\, q_x\,. \tag{70}$$

Depending on the starting point one can construct infinitely many such sequences. For example, $\mathcal{S}$ generates a family of local as well as non-local conserved quantities according to the scheme

$$\begin{aligned} \cdots \quad &\xrightarrow{\mathcal{S}} \frac{1}{2}\, q D_x \left\{ \frac{1}{p_x} D_x \left[\frac{1}{p_x} D_x \left(\frac{q_x}{p_x} \right) \right] \right\} \xrightarrow{\mathcal{S}} -\frac{1}{2}\, \frac{q_x{}^2}{p_x} \\ &\xrightarrow{\mathcal{S}} \tfrac{1}{2}\, q^2\, p_x \xrightarrow{\mathcal{S}} \tfrac{1}{2}\, q\; p_x D_x^{-1} \left[p_x D_x^{-1} (q p_x) \right] \xrightarrow{\mathcal{S}} \cdots\,. \end{aligned} \tag{71}$$

and there are infinitely many different combinations of such sequences. The above expressions for the conserved Hamiltonian densities can be simplified by discarding divergences, however, this form is useful because it suggests the generic term for the conserved quantity obtained by repeated applications of $\mathcal{S}$. The determination of the cardinality of the conserved Hamiltonians which are in involution with respect to Poisson brackets defined by both types of Hamiltonian operators is complicated because of the appearance of arbitrary functions in these Hamiltonian operators.

8.2. Higher flows

Starting with the flow (8) we should obtain higher flows for RHMA_2 through the application of any one of the recursion operators (60), (61), (66) or (67) on the vector field (8).

However, in all of these cases an elementary calculation shows that the resulting higher flow is simply

$$p_t\, q_x - q_t\, p_x = 0$$

the real homogeneous Monge–Ampère equation back again.

9. Symplectic form of RHMA$_2$

The principal geometrical object in the theory of symplectic structure is the symplectic 2-form ω which is closed

$$\mathrm{d}\omega = 0 \tag{72}$$

and by Poincaré's lemma ω can be written as

$$\omega = \mathrm{d}\alpha \tag{73}$$

in a local neighbourhood. On the other hand the Hamiltonian operator maps differentials of functions into vector fields which works in the opposite direction. Thus the statement of the symplectic structure of the equations of motion consists of

$$i_X\omega = \mathrm{d}H \tag{74}$$

which is obtained by the contraction of the symplectic 2-form ω with the vector field $\boldsymbol{X}$ defining the flow.

For systems with finite, and furthermore even number of degrees of freedom, the symplectic 2-form is the inverse of the Hamiltonian structure functions which is the analog of the Hamiltonian operator. However, the generalization of the notion of an inverse to systems with infinitely many degrees of freedom is not immediate. That is, the symplectic 2-form is obtained by integrating the density [16]

$$\omega = \tfrac{1}{2}\,\mathrm{d}u^i \wedge K_{ij}\,\mathrm{d}u^j \tag{75}$$

over the spatial variable, where K is the 'inverse' of J. Given Hamiltonian operator J, its 'inverse' may be defined by

$$J^{ik}K_{kj} = K_{jk}J^{ki} = \delta^i_j \tag{76}$$

but equation (76) is an operator equation which acts on gradients of functions. On the other hand, quite generally there exist Casimir functions which are annihilated by the Hamiltonian operator. Such functions, cf equation (37) above, must be excluded in the definition of the 'inverse' in (76).

Our approach to the construction of Hamiltonian operators is based on the construction of Dirac brackets for systems subject to second class primary constraints. It is evident from (23) that the essential element in the Dirac bracket is the inverse of the matrix of Poisson brackets of the constraints. On the other hand, according to (76) we must invert again to obtain the symplectic 2-form, thus we have simply

$$K_{ik}(x)\delta(x-y) \equiv \{\phi_i(x), \phi_k(y)\} \tag{77}$$

that is, the inverse of the Hamiltonian operator can be obtained directly from the Poisson bracket of second class constraints.

9.1. Symplectic 2-forms

The Hamiltonian operator (29) can be inverted in a straightforward way subject to the provision above:

$$K_\mu = \begin{pmatrix} -\frac{\mu_q q_x}{q} D_x^{-1} - D_x^{-1} \frac{\mu_q q_x}{q} & D_x^{-1} \frac{\mu_q p_x}{q} \\ \frac{\mu_q p_x}{q} D_x^{-1} & 0 \end{pmatrix} \tag{78}$$

or we can immediately read it off from equations (17) and (77). Equation (78) is also a statement of the non-degeneracy of the Hamiltonian operator (29). Hence from (75) and (78) we find the symplectic 2-form

$$\omega_\mu = \frac{\mu_q}{q} \left(p_x \, \mathrm{d}q - q_x \, \mathrm{d}p \right) \wedge \mathrm{d}(D_x^{-1} p) \tag{79}$$

corresponding to the local Hamiltonian operator (29). It can be verified that ω_μ is a closed 2-form by direct calculation. We had found [1] that in the u, q variables the inverse of J_0 is a local operator and this is true for the family of local Hamiltonian operators (29) in general. The symplectic 2-form assumes the simpler expression

$$\omega_\mu = \lambda_{u_x u_x} \left(u_{xx} \, \mathrm{d}q - q_x \, \mathrm{d}u_x \right) \wedge \mathrm{d}u \tag{80}$$

in these variables. By invoking the Poincaré lemma, in a local neighbourhood we can write

$$\omega_\mu = \mathrm{d}\alpha_\mu \qquad \alpha_\mu = \sigma p_x \, \mathrm{d}(D_x^{-1} p) \tag{81}$$

and we note that the coefficient of the 1-form α_μ is also the Casimir (37) for the Hamiltonian operator J_μ. The closure of the symplectic 2-form (79) is equivalent to the satisfaction of the Jacobi identities by the Hamiltonian operator (29).

For the non-local Hamiltonian operator (51) the inverse is given by the local operator

$$K_\kappa = \begin{pmatrix} -\frac{q_x \kappa_{rr}}{p_x^2} D_x - D_x \frac{q_x \kappa_{rr}}{p_x^2} & \frac{\kappa_{rr}}{p_x} D_x \\ D_x \frac{\kappa_{rr}}{p_x} & 0 \end{pmatrix} \tag{82}$$

which also follows from (47) and (77). Then from (75) and (82) the symplectic 2-form appropriate to the non-local Hamiltonian operator is given by

$$\begin{aligned} \omega_\kappa &= \kappa_{rr} \left(-\frac{q_x}{p_x^2} \mathrm{d}p_x + \frac{1}{p_x} \mathrm{d}q_x \right) \wedge \mathrm{d}p \\ &= \kappa_{rr} \, \mathrm{d}r \wedge \mathrm{d}p \end{aligned} \tag{83}$$

so that the verification that ω_κ is closed is immediate. In a local neighbourhood we can write

$$\omega_\kappa = \mathrm{d}\alpha_\kappa \qquad \alpha_\kappa = \kappa_r \, \mathrm{d}p \tag{84}$$

using the Poincaré lemma.

9.2. Symplectic form of equations of motion

With the symplectic 2-forms (79) and (83) we need to check that (74) are satisfied. For this purpose we recall that given a 2-form $\omega = a(v, v_x)\,\mathrm{d}v \wedge \mathrm{d}v_x$ and the vector field $X = m(v, v_x)\,\partial/\partial v$, we have $i_X\omega = (2am_x + ma_x)\,\mathrm{d}v$. If we consider ω_κ with $\kappa = \frac{1}{2}r^2$, from this expression we get

$$
\begin{aligned}
i_X\,\omega_{\frac{1}{2}r^2} &= -\frac{q_x}{p_x}\mathrm{d}q_x - i_{\left(q_x\frac{\partial}{\partial p}\right)}\frac{q_x}{p_x^2}\mathrm{d}p\wedge\mathrm{d}p_x + i_{\left(\frac{q_x^2}{p_x}\frac{\partial}{\partial q}\right)}\frac{1}{p_x}\mathrm{d}p\wedge\mathrm{d}q_x \\
&= -\frac{q_x}{p_x}\mathrm{d}q_x - \left[2\frac{q_x}{p_x^2}q_{xx} + q_x\left(\frac{q_x}{p_x^2}\right)_x - \frac{q_x^2}{p_x^3}p_{xx}\right]\mathrm{d}p - \frac{q_x^2}{p_x^2}\mathrm{d}p_x \\
&= \mathrm{d}\left(-\frac{1}{2}\frac{q_x^2}{p_x}\right) = \mathrm{d}H_{\frac{1}{2}r^2}
\end{aligned}
\tag{85}
$$

where we have discarded a total derivative. Similarly, in order to check that $i_X\,\omega_\mu = \mathrm{d}H_\mu$, we note that given the 2-form $\omega = a(v, v_x, \ldots)\,\mathrm{d}v \wedge \mathrm{d}(D_x^{-1}v)$ and the vector field $X = m(v, v_x, \ldots)\,\partial/\partial v$, we have $i_X\omega = -am\mathrm{d}(D_x^{-1}v) - mD_x^{-1}(a\mathrm{d}v)$. The application to the 2-form (79) yields

$$
\begin{aligned}
i_X\,\omega_\mu &= -q_x D_x^{-1}(\sigma_q p_x\mathrm{d}q) + q_x D_x^{-1}(\sigma_q q_x\mathrm{d}p) \\
&= q\sigma_q(p_x\mathrm{d}q - q_x\mathrm{d}p) = \mathrm{d}H_\mu
\end{aligned}
\tag{86}
$$

using equation (38).

9.3. Witten–Zuckerman 2-form

Time plays a privileged role in Hamiltonian mechanics. While this presents no problem for systems with finitely many degrees of freedom, in field theory it has the disadvantage of non-covariance. In order to remedy this situation Crnković and Witten and Zuckerman [8] have introduced the conserved current 2-form which provides an elegant covariant formulation of Hamiltonian structure. For Monge–Ampère covariance is particularly necessary because, as we noted in the introduction, the choice of time coordinate for RHMA_2 is quite arbitrary.

The simplest way to obtain the Witten–Zuckerman current 2-form ω for the Lagrangian (11) is to first construct the 1-form α which follows from the first variation of the Lagrangian and α is related to ω as in (73). Assuming the equations of motion (6), the first variation reduces to a conservation law

$$
\delta\mathcal{L}_\lambda = {\alpha^t}_{\lambda,\,t} + {\alpha^x}_{\lambda,\,x} \tag{87}
$$

where

$$
\begin{aligned}
{\alpha^t}_\lambda &= \frac{\partial\mathcal{L}_\lambda}{\partial u_t}\delta u + \frac{\partial\mathcal{L}_\lambda}{\partial q_t}\delta q = -q_x\,\lambda_{u_x}\,\delta u + \lambda\,\delta q \\
{\alpha^x}_\lambda &= \frac{\partial\mathcal{L}_\lambda}{\partial u_x}\delta u + \frac{\partial\mathcal{L}_\lambda}{\partial q_x}\delta q = \lambda_{u_x}\,q_t\,\delta u.
\end{aligned}
\tag{88}
$$

Finally, the current 2-form ω_λ for RHMA_2 is given by

$$
\begin{aligned}
{\omega^t}_\lambda &= \delta\alpha^t = \lambda_{u_x u_x}\,(q_x\,\delta u\wedge\delta u_x - u_{xx}\,\delta u\wedge\delta q) \\
{\omega^x}_\lambda &= \delta\alpha^x = \lambda_{u_x u_x}\left(-\frac{q_x^2}{u_{xx}}\,\delta u\wedge\delta u_x + q_x\,\delta u\wedge\delta q\right)
\end{aligned}
\tag{89}
$$

and using the equations of motion we can readily verify that it satisfies

$$\delta\omega^{\alpha} = 0 \qquad \omega^{\alpha}_{\ ,\alpha} = 0 \tag{90}$$

where α ranges over two values t and x. The Witten–Zuckerman 2-form is closed and conserved. For the Lagrangian (43) a similar procedure yields

$$\begin{aligned} \omega^{t}_{\ \kappa} &= \kappa_{rr}\, \delta r \wedge \delta p \\ \omega^{x}_{\ \kappa} &= r\, \kappa_{rr}\, \delta p \wedge \delta r \end{aligned} \tag{91}$$

which also satisfies equations (90). A useful relation in checking the conservation law for the 2-form (91) is the dKdV, or the Riemann equation form of $RHMA_2$

$$r_t = r\, r_x \tag{92}$$

which follows from the results of [18].

We note that the time components of the Witten–Zuckerman 2-forms (89) and (91) for $RHMA_2$ that follow from the Lagrangians (11) and (43) are precisely the symplectic 2-forms (80) and (83), respectively. The use of the notation δ for d follows [8] and is restricted to this section only.

10. Lax pair for $RHMA_2$

We have seen that the recursion operators of section 8 satisfy the Lax equation (62) but these are not useful Lax pairs. We need to cast $RHMA_2$ into the form of a zero-curvature condition [17]

$$U_t - V_x - [U, V] = 0 \tag{93}$$

which is the basic element in the solution of of completely integrable systems using the inverse scattering transform. The zero-curvature condition for the $SL(2, R)$-valued connection 1-form given by the pair

$$U = \begin{pmatrix} \lambda & p_x \\ q\, p_x - \dfrac{\lambda}{p_x^2} p_{xx} - \dfrac{\lambda^2}{p_x} & -\lambda \end{pmatrix} \qquad V = \begin{pmatrix} \lambda \dfrac{q_x}{p_x} & q_x \\ q\, q_x - \dfrac{\lambda}{p_x^2} q_{xx} - \dfrac{\lambda^2}{p_x^2} q_x & -\lambda \dfrac{q_x}{p_x} \end{pmatrix} \tag{94}$$

provides such a formulation of $RHMA_2$. But in this case the potential has a quadratic dependence on the spectral parameter λ which has so far not been considered for an application of inverse scattering techniques. This U, V pair is therefore not immediately amenable to treatment by the method of inverse scattering.

11. Multi-Hamiltonian structure of RMA_2

The infinite classes of Hamiltonian operators we have obtained for $RHMA_2$ reduce to the compatible pair J_0 and J_1 of Hamiltonian operators (30) and (52) when we consider the Hamiltonian structure of RMA_2 with non-zero constant right-hand side in (3). The corresponding pair of symplectic 2-forms are also unchanged and the only modification comes in the conserved quantities. The infinite sequence of conserved Hamiltonians for RMA_2 are those which reduce to the $RHMA_2$ Hamiltonians for $\mu = \frac{1}{2}q^2$ and $\kappa = \frac{1}{2}r^2$ in the limit $K \to 0$. Thus the basic Hamiltonian densities entering into the Lenard–Magri scheme

$$J_0 \, \delta \, H_1^K = J_1 \, \delta \, H_0^K \tag{95}$$

are given by

$$\mathcal{H}_1^K = \frac{1}{2} q^2 p_x + K \, D_x^{-1} p \qquad \mathcal{H}_0^K = -\frac{1}{2 \, p_x} \left({q_x}^2 + K \right) \tag{96}$$

and the infinite sequences of conserved Hamiltonians which are in involution with respect to Poisson brackets defined by J_0 and J_1 are modified for RMA$_2$. For example, by the application of the recursion operator (66) to H_1^K we get

$$\mathcal{H}_2^K = \frac{1}{2 p_x} \left[\left(\frac{q_x}{p_x} \right)_x \right]^2 + \frac{K}{2} \frac{p_{xx}^2}{p_x^5} \tag{97}$$

which, up to a divergence, is the same as the RHMA$_2$ Hamiltonian density in the sequence (71) for $K \to 0$. Repeated application of the recursion operator (66) yields

$$p_t \, q_x - q_t \, p_x = -D_x \left\{ \frac{1}{p_x} D_x \left[\frac{1}{p_x} \ \cdots \ K \ \cdots \ p_x \, D_x^{-1} \left(p_x \, D_x^{-1} \right) \right] \right\} \tag{98}$$

for the RMA$_2$ hierarchy of equations.

The elliptic case of (3) is equivalent to the equation for minimal surfaces while the hyperbolic case corresponds to the Born–Infeld equation [18]. Through the appropriate change of variables the rich multi-Hamiltonian structure of the Born–Infeld equation [19] carries over into RMA$_2$ which includes the Hamiltonian operators J_0 and J_1.

12. Ur-RHMA$_2$

The local and non-local family of Hamiltonian operators for RHMA$_2$ have scalar counterparts for the Ur-RHMA$_2$ equation (4) which can be written as

$$u_t = \frac{k}{u_x} \tag{99}$$

in the form of an evolution equation. With the definition

$$U = \frac{2k}{u_x^2}$$

this equation can be identified as

$$U_t + U \, U_x = 0$$

which is the dispersionless KdV, or Riemann equation. Ur-RHMA$_2$ admits infinitely many conserved quantities

$$\mathcal{H}_n = u_x^n \tag{100}$$

as well as infinitely many local Hamiltonian operators

$$J_\alpha = \frac{k_\alpha}{u_x^\alpha u_{xx}} D_x \frac{1}{u_x^\alpha u_{xx}} \tag{101}$$

provided the various constants are related by

$$n = 2(\alpha + 1) \qquad 2\alpha(\alpha + 1)(2\alpha + 1) k_\alpha = -k$$

so that the equation of motion (99) assumes the form of Hamilton's equations (103). Scalar Hamiltonian operators of this type were first considered by Vinogradov [20]. The family of non-local Hamiltonian operators

$$J_\beta = k_\beta u_x^{\beta} D_x^{-1} u_x^{\beta} \tag{102}$$

which is due to Sokolov [21] is also appropriate to the Ur-RHMA$_2$ equation. In this case the conserved Hamiltonian densities are also given by (100) but now the constants are related by

$$n = -2\beta \qquad 2\beta(2\beta+1)k_\beta = (\beta+1)k$$

and we find that Ur-RHMA$_2$ is cast into Hamiltonian form with

$$u_t = J_\alpha\, \delta H_{2\alpha+2} = J_\beta\, \delta H_{-2\beta} \tag{103}$$

which again results in an infinite set of Hamiltonian structures. In (101) and (102) we have the qq-components of the RHMA$_2$ matrix Hamiltonian operators (29) and (51), respectively.

The recursion operator obtained by the composition of these Hamiltonian operators

$$\mathcal{R}_{\alpha\beta} = \frac{1}{u_x^{\alpha} u_{xx}} D_x \frac{1}{u_x^{\alpha+\beta} u_{xx}} D_x \frac{1}{u_x^{\beta}} \tag{104}$$

can be factored as in (65) with

$$\mathcal{E}_{\alpha\beta} = \frac{1}{u_x^{\alpha}\, u_{xx}} D_x \frac{1}{u_x^{\beta}} \tag{105}$$

resulting in a Sheftel'-type recursion operator for Ur-RHMA$_2$.

13. The RHMA in arbitrary dimension

In order to write RHMA$_n$ as a system of nonlinear evolution equations it will be useful to introduce a compact notation. For this purpose we shall consider the determinants of the $(n-1)\times(n-1)$ matrices

$$\Delta^k \equiv (-1)^{k+1} \det \begin{pmatrix} q_1 & u_{1\,1} & \cdots & \widehat{u_{1\,k}} & \cdots & u_{1\,n-1} \\ q_2 & u_{2\,1} & \cdots & \widehat{u_{2\,k}} & \cdots & u_{2\,n-1} \\ \cdots & \cdots & & \cdots & & \cdots \\ q_{n-1} & u_{n-1\,1} & \cdots & \widehat{u_{n-1\,k}} & \cdots & u_{n-1\,n-1} \end{pmatrix} \tag{106}$$

obtained by deleting the 0^{th} row and k^{th} column in the matrix of second derivatives. The latter is indicated by a hat over the omitted terms. In particular, for $k = 0$ we have the Monge–Ampère operator in $n-1$ dimensions

$$\Delta \equiv -\Delta^0 \neq 0$$

which is a statement of non-degeneracy of RHMA$_n$. The system of evolution equations for RHMA$_n$ is given by

$$\begin{aligned} u_t &= q \\ q_t &= \frac{1}{\Delta} q_i \Delta^i \qquad i = 1, 2, \ldots, n-1 \end{aligned} \tag{107}$$

where $q_i = \partial q/\partial x^i$ and henceforth we shall reserve the index i to range over $n-1$ independent variables while continuing the use of the summation convention over repeated indices. Thus the vector field

$$\boldsymbol{X} = q\frac{\partial}{\partial u} + \frac{1}{\Delta} q_i \Delta^i \frac{\partial}{\partial q} \tag{108}$$

defines the flow for RHMA in n dimensions.

Equations (107) are cast into the form of Hamilton's equations with the Hamiltonian operator

$$\mathcal{J} = \begin{pmatrix} 0 & \dfrac{q}{\mu'\Delta} \\ -\dfrac{q}{\mu'\Delta} & \dfrac{q\Delta^i}{\mu'\Delta^2}D_i + D_i\dfrac{q\Delta^i}{\mu'\Delta^2} \end{pmatrix} \tag{109}$$

where μ is an arbitrary differentiable function of q alone and prime denotes derivative with respect to the argument. The Hamiltonian function is given by

$$\mathcal{H} = \mu\Delta \tag{110}$$

and once again, there exist infinitely many conserved quantities (110) and Hamiltonian operators (109) associated with RHMA in n dimensions. However, this is not the full extent of the Hamiltonian structure of (107) as we have not considered non-local operators, or the dependence of μ on other variables. Concerning the latter point we note that

$$\widetilde{\mathcal{H}} = \mu(q, u_1, u_2, \ldots, u_{n-1})\Delta \tag{111}$$

is also conserved for the system (107). Hence the number of independent variables entering into the arbitrary function μ can be increased considerably with an attendant increase in the number of Hamiltonian operators which is already infinite in (109).

In order to present the symplectic structure of (107) we need the inverse of (109) and subject to the provisions of section 9, we find that it is again a local operator. Then from (75) we get the symplectic 2-form

$$\omega = \frac{\mu'}{q}\left(\Delta^i \mathrm{d}u \wedge \mathrm{d}u_i + \Delta \mathrm{d}q \wedge \mathrm{d}u\right) \tag{112}$$

which can be directly verified to be a closed 2-form. In a local neighbourhood we can write it as the exterior derivative of a 1-form α which is given by

$$\alpha = \sigma\,\Delta\,\mathrm{d}u \tag{113}$$

where σ is again related to μ through (38) and $\sigma\Delta$ is the Casimir for the Hamiltonian operator (109).

14. Geodesic flow for CHMA

The Hamiltonian structure of the geodesic flow for CHMA is very similar to that of RHMA. Semmes [10] has introduced the notion of geodesics on $\mathcal{N}$, the space of smooth real-valued functions on $I \times M$ where I is a real interval. For $F \in \mathcal{N}(I \times M)$ and $(\partial\bar{\partial}F)^n \neq 0$ the vector field

$$X = q\frac{\partial}{\partial F} + n\frac{\left[(\partial\bar{\partial}F)^{n-1} \wedge \partial q \wedge \bar{\partial}q\right]}{\left[(\partial\bar{\partial}F)^n\right]}\,\frac{\partial}{\partial q} \tag{114}$$

defines geodesics on $\mathcal{N}$. The holomorphic exterior derivative is denoted by ∂. Here as well as in the following, it will be understood that volume forms on M enclosed by square parantheses automatically carry the Hodge star operator so that the result is a 0-form. The discussion of the symplectic structure of CHMA by Semmes is based on the Kähler 2-form

$$\frac{1}{2\mathrm{i}}\partial\bar{\partial}F$$

which is not the relevant object that emerges from an examination of the Hamiltonian structure of the flow (114). Our approach to the problem of the geodesic flow for CHMA will be in the framework of dynamical systems with infinitely many degrees of freedom and the resulting symplectic 2-form is given in (118). The advantage of our approach lies in the direct proof it furnishes for the complete integrability of the geodesic flow for CHMA.

The geodesic flow for CHMA satisfies Hamilton's equations

$$\begin{pmatrix} F_t \\ q_t \end{pmatrix} = \boldsymbol{X} \begin{pmatrix} F \\ q \end{pmatrix} = \mathcal{J}_c \delta H_c \tag{115}$$

where $\boldsymbol{X}$ denotes the vector field (114). The Hamiltonian density is given by

$$\mathcal{H}_c = \mu \left[(\partial \bar{\partial} F)^n \right] \tag{116}$$

and the Hamiltonian operator is

$$\mathcal{J}_c = \begin{pmatrix} 0 & \dfrac{q}{\mu' \left[(\partial \bar{\partial} F)^n \right]} \\ \dfrac{-q}{\mu' \left[(\partial \bar{\partial} F)^n \right]} & \mathrm{Re} \left\{ \dfrac{nq \left[\bar{\partial} q \wedge (\partial \bar{\partial} F)^{n-1}}{\mu' \left[(\partial \bar{\partial} F)^n \right]^2} \wedge \partial \ ] - [\partial \wedge \dfrac{nq \bar{\partial} q \wedge (\partial \bar{\partial} F)^{n-1} \ ]}{\mu' \left[(\partial \bar{\partial} F)^n \right]^2} \right\} \end{pmatrix} \tag{117}$$

where $\mu = \mu(q)$ is an arbitrary differentiable function of its argument.

The inverse of Hamiltonian operator (117), subject to the restrictions of section 9, is again a local operator which yields the symplectic 2-form

$$\omega_c = \frac{\mu'}{q} \left\{ \mathrm{Re} \left(\mathrm{d}F \wedge \left[\partial q \wedge (\partial \bar{\partial} F)^{n-1} \wedge \bar{\partial} \right] \mathrm{d}F \right) + \frac{1}{n} \left[(\partial \bar{\partial} F)^n \right] \mathrm{d}q \wedge \mathrm{d}F \right\} \tag{118}$$

and for integrable complex structure ω can be simplified by expressing the exterior derivative in terms of $\partial, \bar{\partial}$. The statement of the symplectic structure of the geodesic flow for CHMA is given by (74). The 2-form (118) is closed as one can show readily by direct calculation. However, it is more instructive to note that by invoking the Poincaré lemma in a local neighbourhood we can write

$$\omega_c = \mathrm{d}\alpha_c \qquad \alpha_c = \frac{1}{n} \sigma(q) \left[(\partial \bar{\partial} F)^n \right] \mathrm{d}F \tag{119}$$

where σ again satisfies equation (38).

There are infinitely many symplectic 2-forms, compatible Hamiltonian operators and conserved Hamiltonians for the geodesic flow for CHMA.

15. Conclusion

We have considered the multi-Hamiltonian structure of various real homogeneous Monge–Ampère equations and found that quite generally they admit *infinitely many* such structures. In particular for RHMA_2 we have shown that there exist infinitely many Hamiltonian operators of both the local and non-local variety. The simplest Hamiltonian operator of the latter type leads to the Kac–Moody algebra of vector fields and functions on the unit circle. For Ur-RHMA_2 we have the scalar version of the RHMA_2 Hamiltonian operators. Finally, we have shown that local Hamiltonian operators are generic to RHMA_n. Thus the real homogeneous Monge–Ampère equation is a system with infinitely many Hamiltonian, or symplectic structures in the theory of integrable systems *in arbitrary dimension*.

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